Effect of Tether Flexibility on the Tethered Shuttle Subsatellite Stability and Control

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This paper investigates the effect of tether flexibility on the (in-plane) stability regions as a function of tether tension control parameters during stationkeeping. It is found that the size of the stability regions for the flexible tether is reduced considerably as compared with that for the rigid massive tether for some control parameters. An alternate optimal control law, which includes additional feedback of the first vibrational mode and its rate, is introduced; the results of stationkeeping simulations show that the transient responses of both the in-plane swing angle and vibrations are improved as compared with previously developed control laws. For retrieval a typical nonlinear control law, which includes the nonlinear feedback of the tether length, in-plane and out-of-plane swing angles and their rates, is developed. Simulation results show that the amplitudes of the in-plane and out-of-plane swing angles could be reduced significantly; also, it is demonstrated that the amplitudes of the tether vibrational modes should be considered for the selection of the control gains.

Introduction

THE dynamics and control of tethered subsatellite systems have been investigated by a host of investigators. A comprehensive survey article was given by Misra and Modi[1] recently. A tether tension control law based on the tether length and length rate for in-plane control was formulated by Rupp[2] and represented a pioneering effort in establishing the feasibility of control using tether tension modulation. Since then, different tension control laws have been developed for deployment, stationkeeping, and retrieval in order to improve system performance. Bainum and Kumar[3] applied linear optimal control theory and introduced a new tether tension control law based on tether length, length rate, in-plane pitch angle and its rate for the case of a taut massless tether. This control system proved to be superior to Rupp's control law during stationkeeping operations. Also, Bainum et al.[4] have investigated the effect of tether flexibility on stability regions for the out-of-plane motion within the linear range. They showed that the unstable parametric resonance of the roll motion can result under specific ratios of tether transverse modal frequency to orbital frequency.

The first objective of this paper is to investigate the effect of tether flexibility on the (in-plane) stability regions previously obtained for the case of the rigid, massless tether of Ref. 3; second, to analyze further the in-plane stability conditions based on Rupp's control law; third, to introduce an optimal control law that now includes feedback of the first in-plane vibrational modal amplitude and its rate; and to compare the transient responses for the three different tension control laws during stationkeeping.

Because the retrieval is basically an unstable procedure, large-amplitude oscillations could be reached for the out-of-plane tether swing motion using a form of Rupp's control law during retrieval. A nonlinear tension control law, which includes nonlinear feedback of the out-of-plane swing angular rate, has been considered by Modi et al.[5] The simulation results showed that during retrieval the amplitudes of the in-plane and out-of-plane rotation could be reduced significantly. An additional objective of the present paper is to develop an extended nonlinear control law, which includes the nonlinear feedback of the length, in-plane and out-of-plane swing angles and their rates, to further improve the transient responses, and to introduce some performance criterion for choosing the control gains in order to reduce the amplitude of the tether transverse vibrations.

System Equations of Motion

Since the Shuttle mass is much greater than the mass of both the subsatellite and the tether, the shift of the center of mass of the entire system from the center of the Shuttle is neglected. The external forces assumed to act on the tethered-subsatellite system are gravitational forces, aerodynamic forces, and tether tension control forces. The system is assumed to move in a circular orbit. However, the altitude variations of the subsatellite caused by the orbital eccentricity are accounted for in the aerodynamic modeling.

The Newton-Euler method has been adopted to develop the system equations of motion. The coordinate system used in the development of the system equations of motion is shown in Fig. 1.

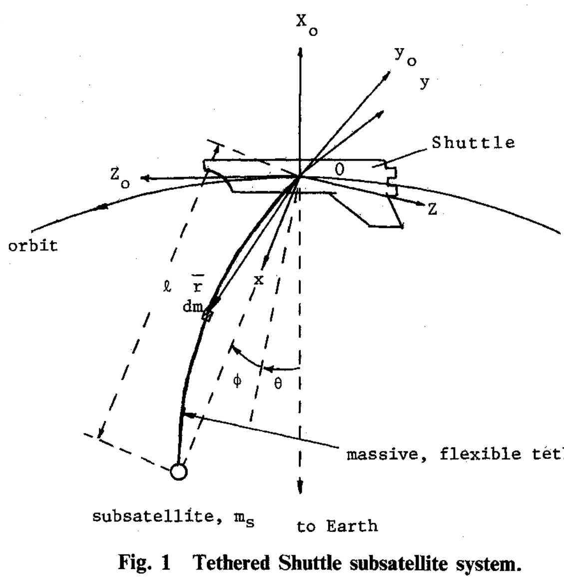

Fig. 1 Tethered Shuttle subsatellite system.

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$OX_0Y_0Z_0$ is an orbit-fixed reference frame centered at the center of mass of the Shuttle, O with OX_0 along the local vertical and OY_0 along the orbit normal opposite to the orbit angular velocity vector. $OXYZ$ is the subsatellite-undeformed tether reference frame (R) with OX along the undeformed tether line.

The angles θ and ϕ are the pitch and roll angle, respectively, which describe the tether in-plane and out-of-plane swing motion. The transformation from $OX_0Y_0Z_0$ to $OXYZ$ is assumed to be given by

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -\cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ \sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & -\cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} \tag{1}$$

The $OXYZ$ frame (R) angular velocity, $\bar{\omega}$, can be written

$$\bar{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} (\dot{\theta} - \omega_c)\sin\phi \\ (\dot{\theta} - \omega_c)\cos\phi \\ \dot{\phi} \end{bmatrix} \tag{2}$$

where ω_c is the orbital angular velocity.

Consider the elemental mass dm whose instantaneous position vector from the center of mass O is $\bar{r}$ (Fig. 1). In the frame R it can be written as

$$\bar{r} = x\bar{i} + \bar{q} \tag{3}$$

where $\bar{q}$ is the elastic displacement vector of dm; the equations of motion for dm can be written as[6,7]

$$\bar{a}dm = L(\bar{q}) + \bar{f}dm + \bar{e}dm$$

and, when viscoelastic forces are to be considered, thus,

$$\bar{a}dm = L(\bar{q}) + L_1(\dot{\bar{q}}) + \bar{f}dm + \bar{e}dm \tag{4}$$

where

$\bar{a}$ = inertial acceleration of dm

$L(\bar{q})$ = elastic forces acting on dm

$L_1(\dot{\bar{q}})$ = viscoelastic forces acting on dm

$\bar{f}$ = gravitational force per unit mass

$\bar{e}$ = external forces acting per unit mass

The gravity force in the frame R is given by[7]

$$\bar{f} = \bar{f}_0 + M\bar{r} \tag{5}$$

where $\bar{f}_0$ is the gravity force at O expressed in the frame R. With the Euler angle sequence shown in Fig. 1, then

$$M = \omega_c^2 \begin{bmatrix} 3\cos^2\phi\cos^2\theta - 1 & -1.5\sin 2\phi\cos^2\theta & 3\cos\phi\cos\theta\sin\theta \\ -3\sin\phi\cos\phi\cos^2\theta & 3\sin^2\phi\cos^2\theta - 1 & -3\sin\phi\cos\theta\sin\theta \\ 3\cos\phi\sin\theta\cos\theta & -3\sin\phi\cos\theta\sin\theta & 3\sin^2\theta - 1 \end{bmatrix} = \omega_c^2 \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{6}$$

The vector equation, Eq. (4), can be written in the frame R as

$$[\bar{a}_0 + \ddot{\bar{r}} + 2\bar{\omega}x\dot{\bar{r}} + \dot{\bar{\omega}}x\bar{r} + \bar{\omega}x(\bar{\omega}x\bar{r}) - \bar{f}_0 - M\bar{r}]dm - L(\bar{q}) - L_1(\dot{\bar{q}}) - \bar{e}dm = 0 \tag{7}$$

where $\dot{\bar{r}}, \ddot{\bar{r}}$ are the velocity and acceleration of dm, respectively, as seen from frame R. By assumption $\bar{f}_0 = \bar{a}_0$, and according to vector algebra,

$$\dot{\bar{\omega}}x\bar{r} + \bar{\omega}x(\bar{\omega}x\bar{r}) - M\bar{r} = [\omega_{ij}]\bar{r} \tag{8}$$

where

$$[\omega_{ij}] = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \omega_c^2 \tag{9}$$

and

$$\omega_{11} = -\phi'^2 - \cos^2\phi[(\theta' - 1)^2 + 3\cos^2\theta] + 1 \tag{10a}$$

$$\omega_{12} = -\phi'' + \sin\phi\cos\phi[(\theta' - 1)^2 + 3\cos^2\theta] \tag{10b}$$

$$\omega_{13} = \theta''\cos\phi - 3\sin\theta\cos\theta\cos\phi \tag{10c}$$

$$\omega_{21} = \phi'' + \sin\phi\cos\phi[(\theta' - 1)^2 + 3\cos^2\theta] \tag{10d}$$

$$\omega_{22} = -\phi'^2 - \sin^2\phi[(\theta' - 1)^2 + 3\cos^2\theta] + 1 \tag{10e}$$

$$\omega_{23} = -\theta''\sin\phi + 3\sin\theta\cos\theta\sin\phi \tag{10f}$$

$$\omega_{31} = -\theta''\cos\phi + 2(\theta' - 1)\sin\phi\phi' - 3\sin\theta\cos\theta\cos\phi \tag{10g}$$

$$\omega_{32} = \theta''\sin\phi + 2(\theta' - 1)\cos\phi\phi' + 3\sin\theta\cos\theta\sin\phi \tag{10h}$$

$$\omega_{33} = -(\theta' - 1)^2 - 3\sin^2\theta + 1 \tag{10i}$$

where $(\ )' = d(\ )/d\tau$ and $\tau = \omega_c t$. Let

$$\bar{r} = (x + u, v, w)^T \tag{11}$$

where u represents the tether's longitudinal elastic displacement, and v and w represent the displacements in orthogonal directions transverse to the OX axis.

To obtain the Rayleigh-Ritz solution, u, v, and w can be expanded in series form in terms of a set of admissible functions

$$u = \sum_n \psi_n(x)A_n(t) \tag{12a}$$

$$v = \sum_n \phi_n(x)B_n(t) \tag{12b}$$

$$w = \sum_n \phi_n(x)C_n(t) \tag{12c}$$

where the functions $\psi_n(x)$ and $\phi_n(x)$ satisfy the geometric boundary conditions. In general,

$$\phi_n(x) = \sin(n\pi x/\ell) \tag{13}$$

For $\psi_n(x)$, two different forms of functions have been adopted. One is the first n eigenfunctions for the longitudinal vibration of a tether of length, ℓ, with a particle mass, m_s, at its end[8,9]

$$\psi_n(x) = \sin(\beta_n x/\ell) \tag{14}$$

where β_n is given by

$$\beta_n \tan\beta_n = m_t/m_s = 1/\mu \qquad (\mu = m_s/m_t) \tag{15}$$

Alternatively, another form of admissible function[10] is given by

$$\psi_n(x) = (x/\ell)^n \tag{16}$$

Function (14) is better than the latter, Eq. (16), because of its more obvious physical significance and the orthogonality properties. Despite this, for system simulation in this paper, both admissible functions have been considered and compared. The question of how significant a role the undeployed tether plays in the longitudinal displacement is still unresolved. Banerjee and Kane[8] and Misra and Modi[9] discussed three different possibilities. In the present paper, it is assumed that the undeployed part of the tether does not undergo any extensional oscillation (no-slip case).

We introduce

$$\begin{aligned}
I_{\phi_n} &= \int_{s,t} \phi_n dm & I_{\psi_n} &= \int_{s,t} \psi_n dm \\
I_x &= \int_{s,t} x dm & H_{\phi\phi} &= \int_{s,t} \phi_n \phi_n dm \\
H_{x\phi_n} &= \int_{s,t} x\phi_n dm & H_{xx} &= \int_{s,t} x^2 dm \\
H_{x\psi_n} &= \int_{s,t} x\psi_n dm & H_{\psi_m\psi_n} &= \int_{s,t} \psi_m\psi_n dm \\
H_{\psi_m\phi_n} &= \int_{s,t} \psi_m\phi_n dm & H_{\phi_m\phi_n'} &= \int_{s,t} \phi_m\phi_n' dm \\
H_{x\phi_n'} &= \int_{s,t} x\phi_n' dm & I_{\psi_n'} &= \int_{s,t} \psi_n' dm, \text{ etc.}
\end{aligned} \tag{17}$$

where the subscript s,t designates the integration over the subsatellite and the tether. With the form of Eq. (16), ψ_n is not orthogonal; with the form of Eq. (14), ψ_n is orthogonal. By using Eqs. (14) and (15), the following results are obtained:

$$H_{x\psi_n} = \ell m_t \sin\beta_n/\beta_n^2, \qquad I_{\psi_n} = m_t/\beta_n \tag{18a}$$

$$H_{\psi_m\psi_n} = \begin{cases} 0 & m \neq n \\ \frac{1}{2}m_t(1 + m_s \sin^2\beta_n/m_t) & m = n \end{cases} \tag{18b}$$

Let

$$\ell = \ell_c + \Delta\ell \tag{19}$$

where ℓ_c is the tether commanded length.

By projecting Eq. (7) along the X axis and integrating the result over the subsatellite and tether, the translation equation of motion can be obtained as

$$\begin{aligned}
&(m_s + m_t)(\Delta\ell'' + \ell_c'') + 2(\theta' - 1)\cos\phi \sum_n I_{\phi_n}(C_n' + C_n\ell'/\ell) \\
&+ \omega_{11}\left(I_x + \sum_m I_{\psi_m}A_m\right) + \omega_{12}\sum_n I_{\phi_n}B_n \\
&+ \omega_{13}\sum_n I_{\phi_n}C_n - 2\phi'\sum_n I_{\phi_n}B_n' + \sum_m I_{\psi_m}A_m'' \\
&+ 2\sum_m I_{\psi_m'}A_m' + \sum_m I_{\psi_m''}A_m = (F_{tx} + F_{ex})/\omega_c^2
\end{aligned} \tag{20}$$

The equations of rotational motion of the system can be obtained by the following operation:

$$\int_{s,t} \bar{r}x\,[\text{Eq. (7)}] = 0 \tag{21a}$$

that is,

$$\int_{s,t} \bar{r}x\{[\ddot{\bar{r}} + 2\omega x\dot{\bar{r}} + [\omega_{ij}]\bar{r}]dm - L(\bar{q}) - L_1(\dot{\bar{q}})\} = \int_{s,t} \bar{r}x\bar{e}dm \tag{21b}$$

By projecting Eq. (21) along the Y and Z axes, respectively, the rotational equations for the pitch (in-plane swing) and roll (out-of-plane swing) are obtained as

$$\begin{aligned}
&\left(\sum_n I_{\phi_n}C_n\right)(\Delta\ell'' + \ell_c'') + 2(\theta' - 1)\cos\phi\Bigg[\left(I_x + \sum_m I_{\psi_m}A_m\right)\ell' \\
&+ \sum_m H_{x\psi_m'}A_m + H_{\phi\phi}\sum_n C_nC_n' \\
&+ \sum_m\left(H_{x\psi_m} + \sum_n H_{\psi_m\psi_n}A_n\right)A_m'\Bigg] - 2(\theta' - 1)\sin\phi \\
&\times\left[\sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)B_n'\right] - 2\phi' H_{\phi\phi}\sum_n C_nB_n' \\
&- 2\sum_n H_{x\phi_n'}C_n' + (\omega_{11} - \omega_{33}) \\
&\times\left[\sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)C_n\right] \\
&+ \omega_{12}H_{\phi\phi}\sum_n C_nB_n + \omega_{13}H_{\phi\phi}\sum_n C_n^2 \\
&- \omega_{31}\left[H_{xx} + \sum_m\left(2H_{x\psi_m} + \sum_n H_{\psi_m\psi_n}A_n\right)A_m\right] \\
&- \omega_{32}\left[\sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)B_n\right] - \sum_n H_{x\phi_n''}C_n \\
&+ \sum_m\left(\sum_n H_{\phi_n\psi_m}C_n\right)A_m'' - \sum_n(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m)C_n'' \\
&= L_{ey}/\omega_c^2
\end{aligned} \tag{22}$$

$$\begin{aligned}
&-(\Sigma I_{\phi_n}B_n)(\Delta\ell'' + \ell_c'') - 2(\theta' - 1)\cos\phi H_{\phi\phi}\sum_n B_nC_n' \\
&-2(\theta' - 1)\sin\phi\left[\sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)C_n'\right] \\
&+ 2\sum_n H_{x\phi_n'}B_n' + 2\phi'\Bigg[H_{\phi\phi}\sum_n B_nB_n' + \left(I_x + \sum_m I_{\psi_m}A_m\right)\ell' \\
&+ \sum_m H_{x\psi_m}A_m' + \sum_n\sum_m H_{\psi_m\psi_n}A_mA_n'\Bigg] \\
&+ (\omega_{22} - \omega_{11})\left[\sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)B_n\right] \\
&- \omega_{12}H_{\phi\phi}\sum_n B_n^2 - \omega_{13}H_{\phi\phi}\sum_n C_nB_n \\
&+ \omega_{21}\left[H_{xx} + 2\sum_m H_{x\psi_m}A_m + \sum_n\sum_m H_{\psi_m\psi_n}A_mA_n\right] \\
&+ \sum_n H_{x\phi_n''}B_n + \omega_{23}\sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)C_n\Bigg] \\
&- \sum_m\left(\sum_n H_{\phi_n\psi_m}B_n\right)A_m'' + \sum_n\left(H_{x\phi_n} + \sum_m H_{\phi_n\psi_m}A_m\right)B_n'' \\
&= L_{ez}/\omega_c^2
\end{aligned} \tag{23}$$

where the terms F_{ex}, L_{ey}, and L_{ez} are contributed by the aerodynamic forces,[11] and F_{tx} is the tension control force.

The nth longitudinal and vibrational mode equations are obtained by the following operations:

$$\int_{s,t} \psi_n [\text{Eq. (7)}]_X = 0 \tag{24}$$

$$\int_{s,t} \phi_n [\text{Eq. (7)}]_Y = 0 \tag{25}$$

$$\int_{s,t} \phi_n [\text{Eq. (7)}]_Z = 0 \tag{26}$$

where $[\text{Eq. (7)}]_X$, $[\text{Eq. (7)}]_Y$, and $[\text{Eq. (7)}]_Z$ mean the projection of vector Eq. (7) on the X, Y, and Z axes, respectively.

In Eq. (24) the longitudinal elastic force term for the tether differential mass, dm, can be expressed as

$$[L(\vec{q})]_x = L(u) = EA \frac{\partial^2 u}{\partial x^2} \mathrm{d}x = EA \sum_m \frac{\partial^2 \psi_m}{\partial x^2} A_m(t)\, \mathrm{d}x \tag{27}$$

and for the subsatellite

$$[L(\vec{q})]_x = 0 - \left(EA \frac{\partial u}{\partial x} \right) \bigg|_{x=\ell} = -EA \sum_m \frac{\partial \psi_m}{\partial x} \bigg|_{x=\ell} A_m(t) \tag{28}$$

Hence,

$$\int_{s,t} -\psi_n L(u) = EA \left[\int_0^\ell -\psi_n \sum_m \frac{\partial^2 \psi_m}{\partial x^2} A_m \, \mathrm{d}x + \psi_n \sum_m \frac{\partial \psi_m}{\partial x} \bigg|_{x=\ell} A_m \right] = \sum_m K_{mn} A_m \tag{29}$$

where

$$K_{mn} = EA \left(\int_0^\ell -\psi_n(x) \frac{\partial^2 \psi_m}{\partial x^2} \mathrm{d}x + \psi_n(\ell) \frac{\partial \psi_m}{\partial x} \bigg|_{x=\ell} \right) \tag{30}$$

After some algebraic manipulations,

$$K_{mn} = \begin{cases} 0 & m \neq n \\ \dfrac{1}{2} \dfrac{EA}{\ell} \beta_m^2 (1 + \mu \sin^2 \beta_m) & m = n \end{cases} \quad \text{for} \quad \psi_m = \sin(\beta_m x/\ell) \tag{31}$$

$$K_{mn} = \frac{EA}{\ell} mn/(m+n-1) \quad \text{for} \quad \psi_m = (x/\ell)^m \tag{32}$$

In Eq. (25), the transverse elastic force term for dm of the tether is

$$[L(\vec{q})]_Y = L(v) = \frac{\partial}{\partial x} \left(F_T \frac{\partial v}{\partial x} \right) \mathrm{d}x \tag{33}$$

where F_T is the tether tension. For steady state, the tether tension is a function of x[9]

$$F_T = 3\omega_c^2 \ell [m_s + m_t(1 - x^2/\ell^2)/2] \tag{34}$$

After substitution of Eqs. (34), (12b), and (13) into Eq. (33) and some sequence of algebraic manipulations, it is obtained that

$$\int_{s,t} -\phi_n L(v) = \left(\sum_n \omega_n^2 B_n \right) H_{\phi\phi} \omega_c^2 \tag{35}$$

where

$$\omega_n^2 = n^2 \pi^2 (1 + 3m_s/m_t) - (3/4) \tag{36}$$

Similarly, in Eq. (26),

$$\int_{s,t} -\phi_n L(w) = \left(\sum_n \omega_n^2 C_n \right) H_{\phi\phi} \omega_c^2 \tag{37}$$

Hence, from Eqs. (24–26), $(NA + 2NB)$ modal equations can be obtained as follows:

$$\begin{aligned} & I_{\psi_n}(\Delta\ell'' + \ell_c'') + 2(\theta' - 1) \cos\phi \sum_m (H_{\psi_n\phi_m} C'_m + H_{\psi_n\phi'_m} C_m) \\ & + \omega_{11} \left[H_{x\psi_n} + \sum_m H_{\psi_n\psi_m} A_m \right] + \omega_{12} \sum_m H_{\psi_n\phi_m} B_m \\ & + \omega_{13} \sum_m H_{\psi_n\phi_m} C_m + \sum_m H_{\psi_n\psi_m} A''_m + \sum_m K_{mn} A_m/\omega_c^2 \\ & + \sum_m K_{mn} A'_m (C_{d_1}/\omega_c) + \sum_m H_{\psi_n\psi'_m} A'_m + \sum_m H_{\psi_n\psi''_m} A_m \\ & - 2\phi' \sum_m H_{\psi_n\phi_m} B'_m = H_{ex}^{(n)}/\omega_c^2 \qquad (n = 1,\ldots,NA) \end{aligned} \tag{38}$$

$$\begin{aligned} & H_{\phi\phi} B''_n - 2(\theta' - 1) \sin\phi H_{\phi\phi} C'_n + 2\phi' \left(I_{\phi_n} \ell' + \sum_m H_{\phi_n\psi_m} A'_m \right) \\ & + \omega_{21} \left[H_{x\phi_n} + \sum_m H_{\phi_n\psi_m} A_m \right] + (\omega_{22} + \omega_n^2) H_{\phi\phi} B_n \\ & + \omega_{23} H_{\phi\phi} C_n + 2 \sum_m H_{\phi_n\phi'_m} B'_m + \sum_m H_{\phi_n\phi''_m} B_m \\ & + C_{d_2} H_{\phi\phi} B'_n/\omega_c = H_{ey}^{(n)}/\omega_c^2 \qquad (n = 1,\ldots,NB) \end{aligned} \tag{39}$$

$$\begin{aligned} & H_{\phi\phi} C''_n + 2(\theta' - 1) \sin\phi H_{\phi\phi} B'_n - 2(\theta' - 1) \cos\phi \\ & \times \left(I_{\phi_n} \ell' + \sum_m H_{\phi_n\psi'_m} A_m + \sum_m H_{\phi_n\psi_m} A'_m \right) \\ & + 2 \sum_m H_{\phi_n\phi'_m} C'_m + \sum_m H_{\phi_n\phi''_m} C_m \\ & + \omega_{31} \left[H_{x\phi_n} + \sum_m H_{\phi_n\psi_m} A_m \right] + \omega_{32} H_{\phi\phi} B_n \\ & + (\omega_{33} + \omega_n^2) C_n H_{\phi\phi} + C_{d_2} H_{\phi\phi} C'_n/\omega_c = H_{ez}^{(n)}/\omega_c^2 \\ & \qquad (n = 1,\ldots,NB) \end{aligned} \tag{40}$$

where the terms $H_{ex}^{(n)}$, $H_{ey}^{(n)}$, and $H_{ez}^{(n)}$ represent the aerodynamic forces, and C_{di}, represents the viscoelastic force coefficients.

System equations (20), (22), (23), (38), (39), and (40) are $(3 + NA + 2NB)$ second-order nonlinear equations, where only first-order nonlinear terms involving ℓ', ℓ'' (important during retrieval/deployment) are indicated.

Stability Analysis for In-Plane Motion with Tension Control Law during Stationkeeping

In Ref. 3, an optimal control law based on tether length, length rate, and in-plane (orbital) swing angle (θ) and its rate was introduced and stability conditions were developed for a rigid, massless tether. In the present paper, the effect of the massive tether's flexibility on the stability conditions will be investigated further. For simplification, the longitudinal displacement and aerodynamics are neglected; only the effect of the transverse vibrational mode will be considered. Because of the decoupling of the in-plane motion from the out-of-plane motion after linearization of the system equations, the linear equations for the in-plane motion during stationkeeping are obtained in the nondimensional variables $\varepsilon(\Delta\ell/\ell_c)$, θ, and $\eta_n(C_n/\ell_c)$ as follows:

$$\varepsilon'' + K_1(2\theta' - 3\varepsilon) - \sum_n K_{2n} \eta'_n = F_{tx}/(m_s + m_t)\ell_c \omega_c^2 + 3K_1 = \Delta f_x \tag{41}$$

$$\theta'' + 3\theta - \sum_n K_{3n}(\eta''_n + 3\eta_n) - K_4 \varepsilon' = 0 \tag{42}$$

$$\eta_n'' + \omega_n^2\eta_n + (C_{d_2}/\omega_c)\eta_n' + ((-1)^n 2/n\pi)(\theta'' + 3\theta)$$
$$+ (4/n\pi)(1-(-1)^n]\varepsilon' = 0 \qquad (43)$$

where

$$K_1 = (m_s + \tfrac{1}{2}m_t)/(m_s + m_t) = (\mu + \tfrac{1}{2})/(\mu + 1) \qquad (44a)$$

$$K_{2n} = 2(1-(-1)^n)/(\mu+1)\pi n \qquad (44b)$$

$$K_{3n} = (-1)^{n+1}/\pi(\mu + \tfrac{1}{3})n \qquad (44c)$$

$$K_4 = (2\mu + 1)/(\mu + \tfrac{1}{3}) \qquad (44d)$$

A linear control law for tether tension levels related to the in-plane variables and their rates is introduced according to

$$\Delta f_x = -(K_\varepsilon \varepsilon + K_{\varepsilon'}\varepsilon' + K_\theta\theta + K_{\theta'}\theta') \qquad (45)$$

where K_ε, $K_{\varepsilon'}$, K_θ, and $K_{\theta'}$ are the control gains (constants). In state-variable format, Eqs. (41–43) can be written as

$$X' = AX \qquad (46)$$

where

$$X^T = (\theta\theta'\varepsilon\varepsilon'\eta_1\eta_1',\ldots,\eta_{NB}\eta_{NB}')$$
$$A = \{a_{ij}\} \qquad (47)$$

and a_{ij} are functions of the system parameters and the control gains. As a first step, we consider only the first vibrational mode; hence, the system characteristic equation is given by

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \qquad (48)$$

where a_i are functions of a_{ij}. The Routh-Hurwitz criteria can be used to develop the stability conditions.

Let $x_1 = K_\theta, y_1 = K_{\theta'}$ for the control gains K_ε, $K_{\varepsilon'}$, x_1, and y_1, the following stability conditions can be obtained:

$$K_{\varepsilon'} > a_{66} (\text{if } C_{d2} = 0 \quad \text{then} \quad a_{66} = 0) \qquad (49a)$$

$$K_\varepsilon > 3K_1 \qquad (49b)$$

$$D_{21}x_1 + D_{22}y_1 + D_{23} > 0 \qquad (49c)$$

$$D_{31}x_1^2 + D_{32}x_1y_1 + D_{33}y_1^2 + D_{34}x_1 + D_{35}y_1 + D_{36} > 0 \qquad (49d)$$

$$D_{41}x_1^3 + D_{42}x_1^2y_1 + D_{43}x_1y_1^2 + D_{44}y_1^3 + D_{45}x_1^2 + D_{46}x_1y_1$$
$$+ D_{47}y_1^2 + D_{48}x_1 + D_{49}y_1 + D_{410} > 0 \qquad (49e)$$

$$D_{51}x_1^4 + D_{52}x_1^3y_1 + D_{53}x_1^2y_1^2 + D_{54}x_1y_1^3 + D_{56}x_1^3$$
$$+ D_{57}x_1^2y_1 + D_{58}x_1y_1^2 + D_{59}y_1^3 + D_{510}x_1^2 + D_{511}x_1y_1$$
$$+ D_{512}y_1^2 + D_{513}x_1 + D_{514}y_1 + D_{515} > 0 \qquad (49f)$$

where D_{2i}, D_{3i}, D_{4i}, and D_{5i} are complicated functions of the system parameters and control gains K_ε and $K_{\varepsilon'}$.

For given K_ε and $K_{\varepsilon'}$, which satisfy conditions (49a) and (49b), the regions where conditions (49c–49f) are satisfied can be obtained in the control gain space (x_1,y_1). When more vibrational modes are to be considered similar stability regions can be obtained. One of the typical results is shown in Fig. 2.

The results show that the stability conditions involving the control parameters K_ε and $K_{\varepsilon'}$ for the flexible tether are similar to those for the rigid tether. However, the stability region dependent on the control parameters K_θ and $K_{\theta'}$ for the flexible tether is reduced as compared with that for the rigid tether. Also, the size of the stability regions for the system, where two or three vibrational modes are to be considered, is reduced as compared with that for the system where only one mode is to be considered. However, the size of the stability region when two modes are retained is nearly the same as that for the case where three modes are included. Finally, with tether natural damping ($C_{d_2} \neq 0$) the stability region can be enlarged in the K_θ,$K_{\theta'}$ parameter space (the case of one vibrational mode is included, curve 5).

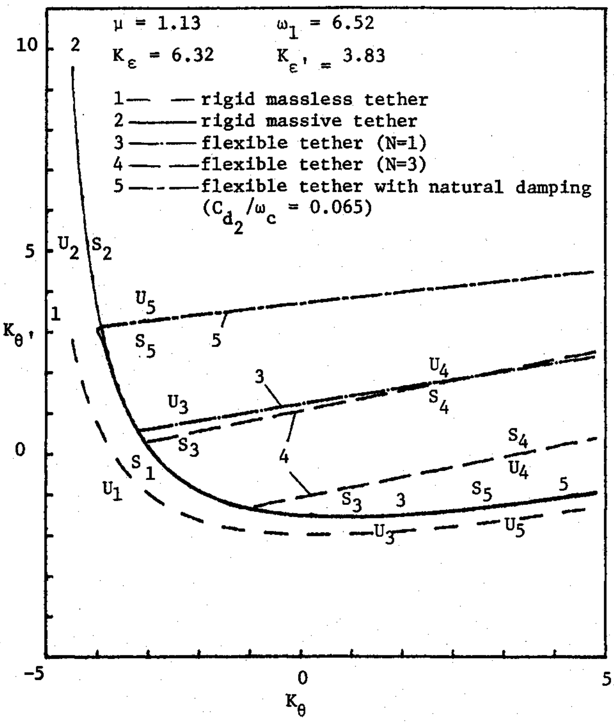

Fig. 2 Comparison of stability regions.

For Rupp's control law ($x_1 = y_1 = 0$), it was obvious that the stability conditions in the control parameter space (K_ε,$K_{\varepsilon'}$) are given by (assume $C_{d_2} = 0$)

$$K_{\varepsilon'} > 0, \qquad K_\varepsilon > 3K_1, \qquad D_{23}(K_\varepsilon,K_{\varepsilon'}) > 0$$
$$D_{36}(K_\varepsilon,K_{\varepsilon'}) > 0, \qquad D_{4\,10}(K_\varepsilon,K_{\varepsilon'}) > 0$$
$$D_{5\,15}(K_\varepsilon,K_{\varepsilon'}) > 0 \qquad (50)$$

It is demonstrated that the conditions of (50) are equivalent to the conditions

$$K_{\varepsilon'} > 0 \qquad K_\varepsilon > 3K_1 \qquad (51a)$$

and

$$\omega_1 > [6/(\mu + 1/3)]^{1/2}/\pi \qquad (51b)$$

for a single transverse tether mode [from numerical results (not shown) the inclusion of additional modes will not greatly affect this stability boundary in parameter space]. According to Eq. (36), condition (51b) is always satisfied for any μ. Hence, the stability conditions are (51a), which are the same as those for the massive, rigid tether.

Optimal Control Including First Transverse Vibrational Mode Feedback during Stationkeeping

The linearized equations, including the first vibration mode for in-plane motion, can be written in the following state-variable format:

$$X' = A_1X + BU \qquad (52)$$

Table 1 ORACLS subroutines [12]

K_ε	$K_{\varepsilon'}$	K_θ	$K_{\theta'}$	K_{η_1}	$K_{\eta_1'}$
5.0	4.0	1.81	0.45	7.16	−0.85

Table 2 System parameters

	Satellite	Tether
Mass, kg	550	485 (100 km length)
Drag area, m^2	2.01	
Diameter, mm		1.91
Drag coefficient	2.2	2.2 (normal) 0.2 (axial)
Axial stiffness, (AE)N		61,645

where $X^T = [\theta\theta'\varepsilon\varepsilon'\eta_1\eta_1']$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & a_{24} & a_{25} & a_{26} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2K_1 & 3K_1 & 0 & 0 & K_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a_{64} & a_{65} & a_{66} \end{bmatrix} \tag{53}$$

$$B^T = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0] \tag{54}$$

It has been verified that this system is still controllable using only tether tension control, and it is also observable with the measurements of ε,ε' and θ,θ' or only ε,ε'. Thus, the first flexible modal state variables (η_1,η_1') are available by estimation with the measurement of ε,ε', or ε,ε' and θ,θ', or, possibly, by some additional measurements of tether transverse displacement at selected points along the tether.

In order to improve the transient response of the system, a flexible tether optimal state feedback control law based on ε,ε', θ,θ', and η_1,η_1' has been introduced. The optimal control U, which minimizes the performance index

$$J = \int_0^\infty (X^TQX + U^TRU)\,\mathrm{d}t \tag{55}$$

is given by

$$U = -(R^{-1}B^TP)X = -KX \tag{56}$$

where Q is a positive semidefinite state penalty matrix, R a positive definite control penalty matrix, and P the positive definite solution to the steady-state Riccati matrix equation.

$$-PA_1 - A_1^TP + PBR^{-1}B^TP - Q = 0 \tag{57}$$

For a set of typical parameters—$\mu = 550/485 = 1.13$, $\ell_c = 100$ km, $R = 5$, and $Q = 10\delta_{ij}$—the optimal gains are given by using subroutines from ORACLS[12] in Table 1.

The results of system simulation show that the transient response can be improved significantly with this optimal control law. It will be discussed in the next section.

System Simulation and Results

The nonlinear and coupled differential equations [(20), (22), (23), (38–40)] were integrated numerically to investigate the system responses for different control laws and disturbance forces during stationkeeping, retrieval, and deployment.

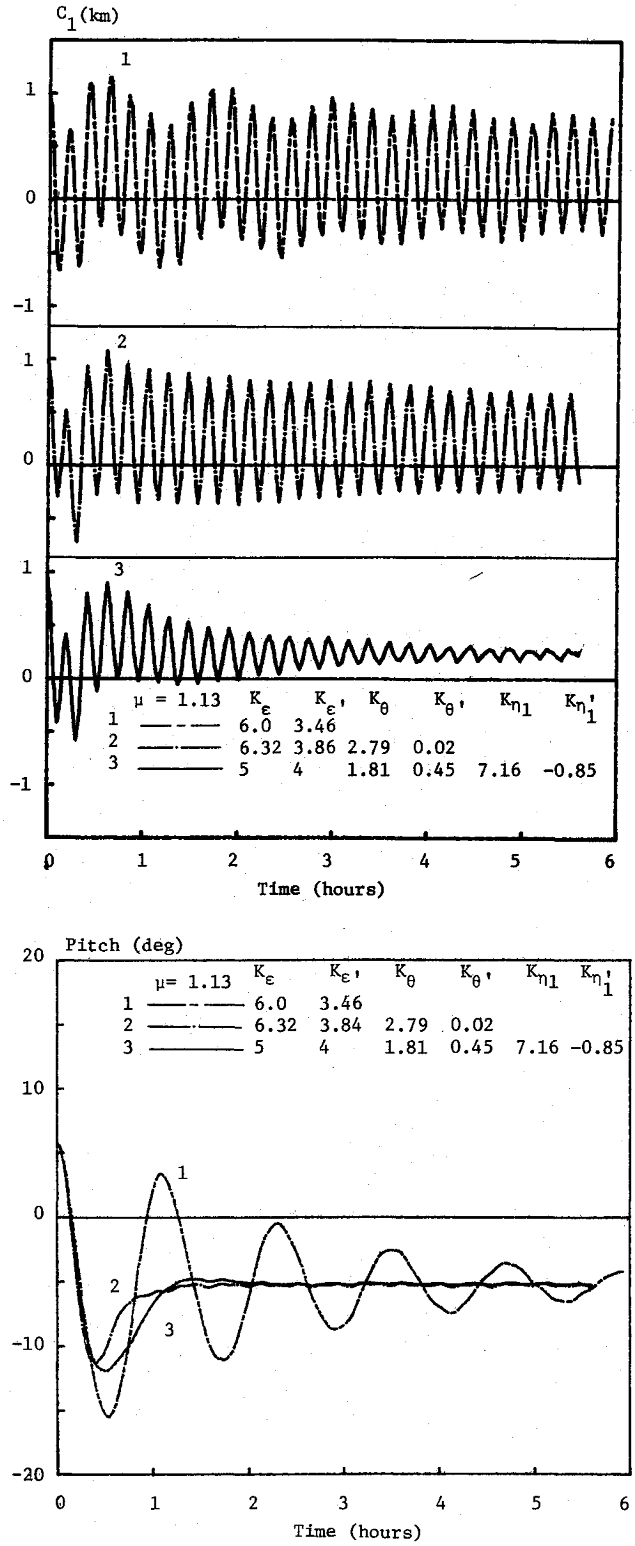

Fig. 3 Comparison of transient responses for three linear control laws with aerodynamics.

The system parameters in Table 2 are adopted as provided by the 1987 NASA/PSN Tether Applications Simulation Working Group.

Stationkeeping

A typical comparison of transient responses for the three different control laws with aerodynamics is shown in Fig. 3. It is seen that Rupp's control law could be used to control the in-plane swing angle successfully (Fig. 3b, curve 1) during stationkeeping, but it is not very effective for damping the

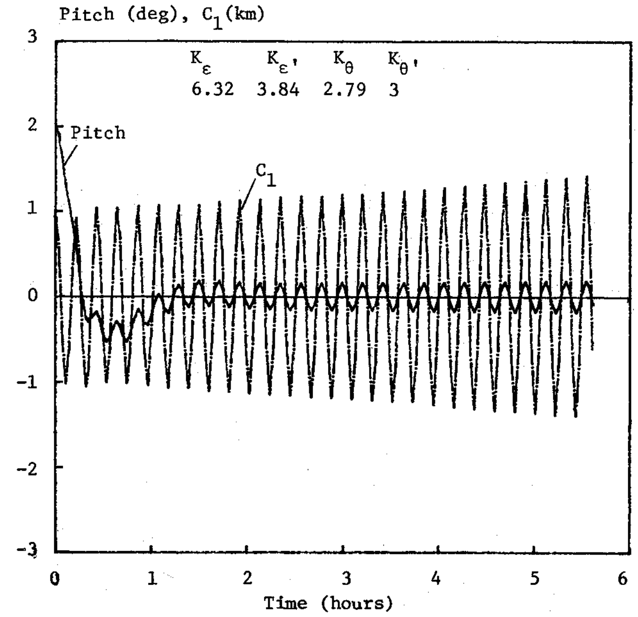

Fig. 4 Stationkeeping (unstable case) using linear control.

in-plane vibrations (Fig. 3a, curve 1). The transient responses for Bainum's optimal control law based on $\varepsilon, \varepsilon'$ and θ, θ' are faster than those for Rupp's control law (Figs. 3a and 3b, curve 2). The transient response can be further improved by including the state feedback of the first vibrational mode into the optimal control law (Figs. 3a and 3b, curve 3). Considerable improvement is noted in both the in-plane swing angle and, especially, the first vibration modal amplitude.

For verifying the results of the stability analysis, a typical unstable transient response for a control law, whose control gains are chosen intentionally in the unstable region, is shown in Fig. 4.

Nonlinear Control Law for Retrieval

The retrieval is basically an unstable procedure. With the exponential commanded length $\ell_c = \ell_0 e^{-t/p}$, Rupp's tether tension linear control strategy based on length and length rate could be used for retrieval, but large-amplitude out-of-plane-swing oscillations, which approach 60 deg for a 6-h retrieval (from 100 to 1 km), could be noticed. Meanwhile, it is found that there is not much improvement for the out-of-plane swing motion by using other linear control laws. Modi et al. introduced some nonlinear control strategies for a tether length rate control law[13] and a tension control law[5] in the form of $T = K_\ell \ell + K_{\ell'} \ell' + K_{\phi'^2} \phi'^2$. The results showed that the amplitude of the out-of-plane swing angle could be significantly reduced; however, during our simulation, it was found that it is better if the control gain $K_{\phi'^2}$ is dependent proportionally on the tether length in order to keep the tether taut because the tether tension is approximately proportional to the tether length. In addition, it is observed that the nondimensional vibrations η_n are increased gradually during retrieval, and as the gain $K_{\phi'^2}$ is increased (absolute value) the vibrations are also increased; thus, the amplitudes of the vibrations should be considered for choosing control gains. In the present paper, an additional nonlinear tension control law is to be introduced in the following form:

$$F_{tx} = -(m_s + m_t)\omega_c^2 [F_c + FK_1 \Delta\ell + FK_2 \Delta\ell' + \underline{\ell}(F\underline{K_3}\theta^2 + F\underline{K_4}\phi^2 + FK_5\phi'^2)] \tag{58}$$

in order to improve further the performance of the system. Note that the terms underlined were not included in Ref. 5.

It is difficult to use analytical methods to derive control gains for such nonlinear equations. Hence, for choosing the control gains, the performance criterion will be introduced.

$$J_1 = \left[\int_0^{T_f} (\theta^2 + \phi^2)\,\mathrm{d}t\right] \Big/ T_f \tag{59}$$

$$J_2 = \left\{\int_0^{T_f} [(B_1/\ell)^2 + (C_1/\ell)^2]\,\mathrm{d}t\right\} / T_f \tag{60}$$

$$J = J_1 + Q_2 J_2 \tag{61}$$

where T_f is the desired retrieval time and Q_2 a weighting coefficient. By simulation, some of the better control gains (resulting in a smaller J) can be obtained for specific initial conditions and essentially the same retrieval times, where the tether remains taut throughout the operation. One set of typical control gains is given in Table 3. Under conditions of approximately 6.2-h retrieval from 100 km to 1 km and with $\theta(0) = \phi(0) = 5.7$ deg, $Q_2 = 1$, according to simulation results, the following conclusions could be reached: first, the amplitudes of the swing angles θ and ϕ, especially ϕ, are reduced greatly (Fig. 5) as compared with Rupp's control law, and the transient responses are also better than those based on the control law in Ref. 5 (Fig. 6); second, the nondimensional vibration η_n is increased gradually, and, as the gain, FK_5, is increased (absolute value), the responses of θ and ϕ are improved (J_1 decreased), but the amplitude of the transverse vibrations B and C (so J_2) is increased. Therefore, J_2 and the

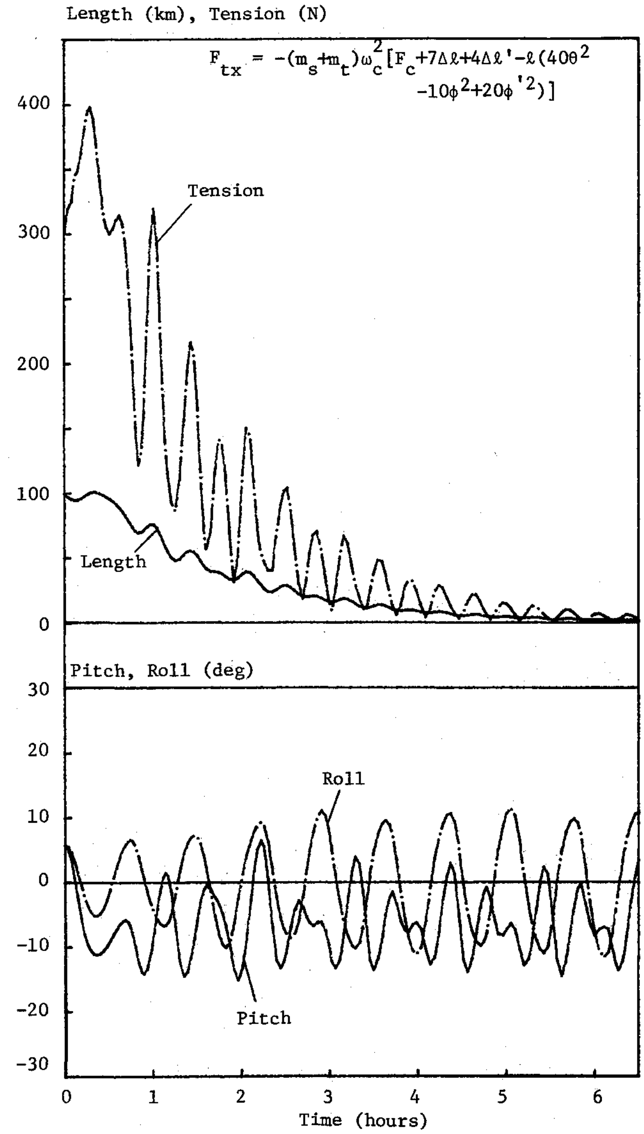

Fig. 5 Retrieval with nonlinear control law and aerodynamics.

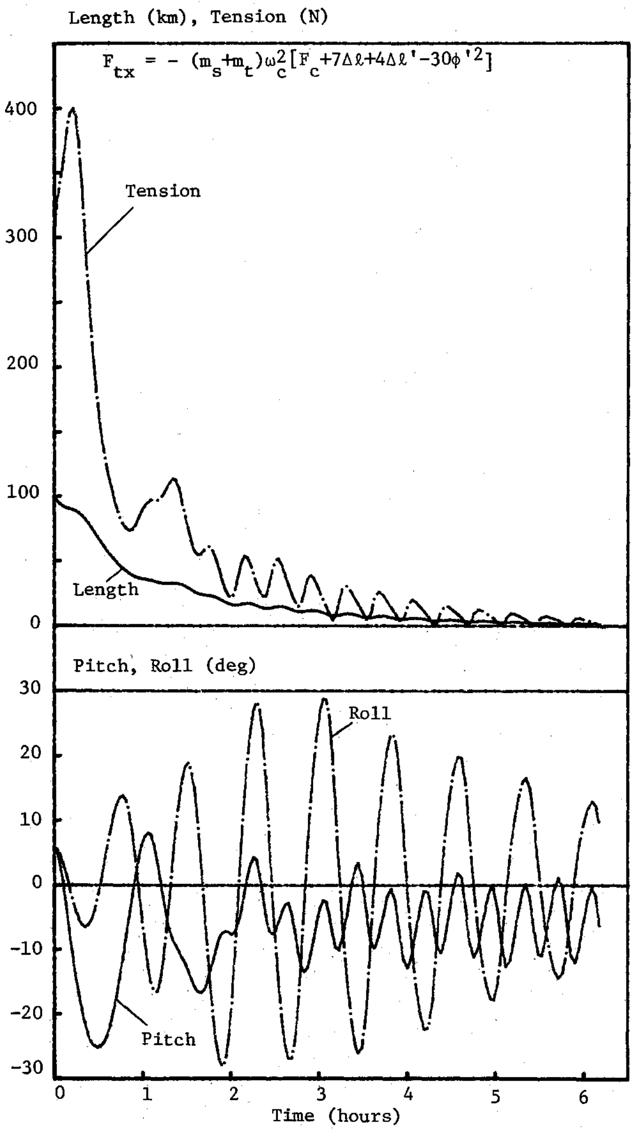

Fig. 6 Retrieval with nonlinear control law and aerodynamics.

Table 3 Typical control gains

FK_1	FK_2	FK_3	FK_4	FK_5
7	4	−40	10	−20

weight coefficient, Q_2, should be included in the system performance index, J, in order to reduce the vibration amplitudes.

By numerical means, on a case-by-case basis, it is seen that the nonlinear control law remains effective in the nearby region of initial conditions; however, the vibrational amplitudes are increased gradually during retrieval. As a result of large in-plane vibrational amplitude (over 1 km), at the end of a retrieval it is recommended that after half of the retrieval this operation should be stopped and a stationkeeping procedure carried out. Simulation results show that, by the end of retrieval, the first vibrational amplitude could be reduced to 0.1 km, using the previously introduced stationkeeping control law for a 3-h stationkeeping at 20 km length, before the retrieval is again resumed.

Simulation results show that the damping of the higher vibrational modes is not very effective with the present control law. However, the results indicate that most of the vibrational displacement is attributed to the first vibrational mode during retrieval. An extension to the present study could consider the damping of the higher vibrational modes.

Conclusions

1) The tether flexibility significantly affects the stability conditions, which depend on the control gains of the in-plane swing angle and its rate. The size of the stability regions for the flexible tether is reduced considerably as compared with that for the rigid, massive tether; the size of the stability regions when two transverse modes are included in the model is nearly the same as that for the case where three modes are included.

2) Rupp's law is not very effective for damping in-plane tether vibrations, but an optimal tension control law, including the feedback of the first vibrational mode, can be used to damp both tether vibrations (especially the first vibrational mode) and the in-plane swing angle effectively.

3) A nonlinear tension control law, which includes nonlinear feedback of the tether length, and in-plane and out-of-plane swing angles and their rates could be used successfully for retrieval. The amplitudes of the in-plane and out-of-plane swing angles are reduced significantly. Since the nondimensional vibrations are increased gradually during retrieval, a vibrational performance index should be considered for choosing the nonlinear control gains.

4) Because of the large in-plane vibrational amplitude (over 1 km), at the end of a retrieval it is recommended that after one-half of the retrieval this operation should be stopped and a stationkeeping procedure carried out, before the retrieval is again resumed.

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